CARTAN TORI AND ADE CLASSIFICATION OF TWO-DIMENSIONAL TOPOLOGICAL PHASE TRANSITIONS

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I. INTRODUCTION

The topological phase transition (TPT) or the Berezinskii-Kosterlitz-Thouless (BKT) phase transition (PT) takes place in two-dimensional systems with order parameter $\psi = e^{2\pi i\varphi} \in \mathcal{M} = S^1$. Among them are XY-model, superconductors, bose-liquids and many other systems [2, 18, 14, 12, 21]. XY-model on a lattice is defined as follows

$$\mathcal{Z}_{XY} = \sum \exp(-\beta\mathcal{H}),$$

$$\mathcal{H} = -\frac{1}{2}\sum_{<i,j>} J(\psi_i\psi_j^* + c.c.) = -J\sum_{<i,j>}\cos(\varphi_i - \varphi_j). \tag{1}$$

Its continuous variant is a nonlinear σ-model (NSM) on S^1

$$\mathcal{Z}_{NS} = \int D\varphi e^{-\mathcal{S}[\varphi]}, \quad \mathcal{S}_{NS} = \frac{1}{2\alpha}\int |\partial\psi|^2 d^2x = \frac{1}{2\alpha}\int (\partial\varphi)^2 d^2x, \tag{2}$$

$$\alpha \simeq T/2J.$$

A circle S^1 has a nontrivial homotopic group π_1

$$\pi_1(S^1) = \mathbb{Z}. \tag{3}$$

Due to this fact the topologically stable excitations, vortices, are possible in these systems. One vortex solution has a form (at large distancies)

$$\varphi(\mathbf{x}) = \frac{1}{\pi}\arctan\frac{y}{x}, \quad \mathbf{x} = (x, y) \in \mathbb{R}^2, \tag{4}$$

An account of vortices means that theory must be considered on the covering space $\mathbb{R}$ of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The energy of one "vortex" is logarithmically divergent, but the energy E_N of N vortices with the full topological charge $e = \sum_{i=1}^N e_i = 0$ is finite and equals

$$E_N = \frac{2\pi}{2\alpha}\sum_{i\neq k}^N e_i e_k \ln\frac{|\mathbf{x}_i - \mathbf{x}_k|}{a} + C(a)\sum_i^N e_i^2, \tag{5}$$

here $C(a)$ is some nonuniversal constant, determining "self-energy" (or core energy) of vortices and depending on type of core regularization.

The partition function $\mathcal{Z}_{XY}$ can be approximated by product of two partition functions [14, 12]

$$\mathcal{Z}_{XY} \simeq \mathcal{Z}_{sw}\mathcal{Z}_{CG}, \tag{6}$$

where $\mathcal{Z}_{CG}$ is the grand partition function of dilute Coulomb gase (CG) of topological exitations with minimal charges $e = \pm 1$ and $\mathcal{Z}_{sw}$ is the partition function of *free* "spin-waves". $\mathcal{Z}_{CG}$ can be represented, in its turn, in the form of effective field theory with sine-Gordon (SG) action $\mathcal{S}_{SG}$ [12, 21]

$$\mathcal{Z}_{CG} = \mathcal{Z}_{SG} = \int D\phi e^{-\mathcal{S}_{SG}[\phi]}, \tag{7}$$

$$\mathcal{S}_{SG} = \int \left[\frac{1}{2\alpha}(\partial\phi)^2 - 2\mu^2\cos\phi\right] d^2x$$

The TPT takes place in the system of vortices and can be described by the effective SG theory. This system has two different phases:

1) **high-T phase**: plasma-like, massive, with finite correlation length with essential singularity at T_c [14]

$$\xi \sim a \exp(A\tau^{-\nu}), \quad \nu = \frac{1}{2}, \tag{8}$$

$$\tau = \frac{T - T_c}{T_c} \to 0, \quad T_c \approx \pi J;$$

2) **low-T phase**: dielectric, massless, with infinite correlation length and algebraically falling correlations [20, 2, 18].

Question:

Are there any possible generalizations on systems with more complicate group π_1 and other types of critical behaviour?

The answer is nontrivial. The simplest generalization of circle S^1 is a torus T^n with the homotopy group

$$\pi_1(T^n) = \bigoplus_{i=1}^{n} \mathbb{Z}_i = \mathbb{Z}^n, \tag{9}$$

where i-th component describes maps of the boundary S^1 into i-th circle of T^n.

The maps into different components cannot be transformed into or annihilate each other. Consequently, one can introduce in $\pi_1(T^n)$ and in space of corresponding topological charges a vector structure: a vector basis and a metric. In case of T^n it is an usual euclidean structure with canonical basis $\{\mathbf{e}_i\}$ and metrics $g_{ik} = \sum_{a=1}^{n} e_i^a e_k^a = \delta_{ik}$. Then the topological charges, corresponding to different S_i^1, do not interact! Thus, the theories with $\mathcal{M} = T^n$ simply replicate the case $\mathcal{M} = S^1$ and reduce to it.

Moreover, all tori $T_L = \mathbb{R}^n / \mathbb{L}$,

$$\mathbb{L} = \sum_{i=1}^{n} n_i \mathbf{e}_i, \ n_i \in \mathbb{Z}_i, \ \mathbf{e}_i \in \{\mathbf{e}_i\}_L, \quad g_{ik} = \sum_{a=1}^{n} e_i^a e_k^a, \tag{10}$$

where $\mathbb{L}$ is n-dimensional lattice in $\mathbb{R}^n$ ($\{\mathbf{e}_i\}_L$, $i = 1, ..., n$ forms a basis of lattice $\mathbb{L}$), and g_{ik} is an effective metric determined by the theory action, reduce in the same sence to the case S^1 also [4]. It means that

the torus properties have some hardness (or stability):

the smooth deformations in general position do not change them.

This fact is connected with corresponding deformation of effective metric in space of topological charges. In terms of critical properties of TPT it means that for all systems, having such tori as a vacuum space, the TPT will have the same critical properties [4].

Proposal:

For obtaining nontrivial generalizations one must consider NSM on the degenerated tori, in particular, on the Cartan tori T_G of the simple compact Lie groups G.

II. CARTAN SUBGROUP AND DEGENERATED TORI

The Cartan torus T_G, the maximal abelian subgroup of group G, consists of elements

$$\mathbf{g} = e^{2\pi i(\mathbf{H}\,\boldsymbol{\phi})},\ \mathbf{H} = \{H_1, ..., H_n\} \in \mathcal{C},\ [H_i, H_j] = 0,$$

where n is a rank of G, $\mathcal{C}$ is a maximal commutative Cartan subalgebra of the Lie algebra $\mathcal{G}$ of the group G. It is assumed here that $(\mathbf{H}\,\boldsymbol{\phi})$ is an usual euclidean scalar product. All H_i can be diagonalized simultaneously. Their eigenvalues, the weights $\mathbf{w}$ (or quantum numbers), depend on the concrete representation of G and $\mathcal{C}$.

All possible weights $\mathbf{w}$ of the simply connected group G (or of the universal covering group $\tilde{G}$ of the non-simply connected group G) form a lattice of weights L_w. In this basis all H_i (and any element $\mathbf{g} \in T_G$) get a diagonal form

$$\mathbf{g}_\tau = diag(e^{2\pi i(\mathbf{w}_1\,\boldsymbol{\phi})}, ..., e^{2\pi i(\mathbf{w}_p\,\boldsymbol{\phi})}) \tag{11}$$

The main differences of this form from the usual representation of T_L type tori are:

1) a dimension of diagonal matrices coincides with dimension p of τ-representation, which is usually larger, than rank of G;

2) the set of weights $\{\mathbf{w}\}_\tau$ has a discrete Weyl (or crystallographic) symmetry, which results in the next two properties

$$\sum_{a=1}^{p} \mathbf{w}_a = 0, \quad g_{ik} = \sum_{a=1}^{p} w_i^a w_k^a = B_\tau \delta_{ik}, \tag{12}$$

where constant B_τ depends on representation. Now the effective metrics g_{ik} is proportional to the euclidean one.

Tori T_G can be considered as appropriately constrained (reduced) torus $T^N = T_{U(N)}$ with large enough N. They give the examples of degenerated tori $\mathcal{T}_L$, which we define, in general case, by diagonal matrices of form (11), containing all minimal vectors of lattice $\mathbb{L}$. A dimension of these matrices p equals to the number of all minimal vectors of lattice $\mathbb{L}$, which is usually larger than dimension of lattice.

In the form (11) all $\mathbf{g} \in T_G$ are periodic with a lattice of periods L^t_τ, inverse to the lattice L_τ, generated by weights $\mathbf{w}_a (a = 1, ..., p)$ of τ-representation. L^t_τ forms a set of all topological charges of τ-representation of T_G.

The lattice L^t_τ satisfies the next restriction

$$L_{w^*} \supseteq L^t_\tau \supseteq L_v.$$

where L_{w^*} is a weight lattice of dual group G^*, L_v is a lattice of dual roots. For $\tau = min$ a lattice $L^t_\tau = L_v$, for $\tau = ad$ a lattice $L^t_\tau = L_{w^*}$. The lattices L_v and L_{w^*} differ by a factor, which is isomorphic to the centre Z_G of group G

$$L_{w^*}/L_v = Z_G.$$

Thus the set of minimal topological charges $\{\mathbf{q}\}_\tau$ can vary from the set of minimal vectors of the weight lattice till that of the dual root lattice. All possible cases are determined by subgroups of the centre Z_G. For groups G with $Z_G = 1$ the lattices L_v and L_{w^*} coincide, i.e. they are self-dual ($G = E_8$).

When $L^t_\tau = L_w$ (or $L^t_\tau = L_{w^*}$) all weights (i.e. quantum numbers) of group G (or G^*) can be reproduced as the vector topological charges of vortices!

Analogously, a lattice of topological charges L^t_L of degenerated torus $\mathcal{T}_L$ belongs to a reciprocal lattice $\mathbb{L}^{-1}$. Since the integer-valued lattices L also belong to their own inverse lattices $\mathbb{L}^{-1}$, their lattices of topological charges, in general, are even larger then $\mathbb{L}$: $L^t_L \supseteq \mathbb{L}$. For this reason they can contain *fractional* (in this basis) topological charges. Only for degenerated tori $\mathcal{T}_L$, connected with self-dual lattices, L^t_L exactly coincide with L. Consequently, for tori associated with the integer-valued lattices all their "quantum numbers" always have a topological interpretation.

III. NS-MODELS ON T_G, DUALITY AND EFFECTIVE THEORIES

The euclidean two-dimensional NSM on T_G , have the following form

$$\mathcal{S} = \frac{1}{2\alpha} \int d^2x Tr_\tau(\mathbf{t}_\nu^{-1}\mathbf{t}_\nu) = \frac{(2\pi)^2}{2\alpha} \int d^2x Tr_\tau(\mathbf{H}\,\boldsymbol{\varphi}_\nu)^2 \tag{13}$$

$$= \frac{(2\pi)^2}{2\alpha} B_\tau \int d^2x (\,\boldsymbol{\varphi}_\nu)^2,$$

where $\boldsymbol{\varphi}_\nu = \partial_\nu \boldsymbol{\varphi}$, $\nu = 1, 2$. An including of a factor B_τ into trace Tr_τ gives a canonical euclidean metric in space of topological charges.

These theories are invariant under direct product of right (R) and left (L) groups $N_G^{R(L)}$, which are a semi-direct product of T_G and W_G

$$N_G = T_G \times W_G. \tag{14}$$

The group $N_G \in G$ is called a normalizator of T_G and is a symmetry group of torus T_G, thus the theories (13) can be considered also as chiral NSM on group G with symmetry breaking $G \searrow N_G$. The NSM on T_G have properties analogouos to those of XY -model:

1) a zero beta-functions $\beta(\alpha)$ due to flatness of T_G;

2) non-trivial homotopy group π_1 and corresponding vortex solutions with topological charges $\mathbf{q} \in L^t_\tau$.

In quasi-classical approximation (or in low T expansion) a partition function of the σ -model on T_G can be represented as a grand partition function of classical neutral Coulomb gas of vortices with topological charges $\mathbf{q}_i \in \{\mathbf{q}\}_\tau$

$$\mathcal{Z} = \mathcal{Z}_0 \mathcal{Z}_{CG}, \quad \mathcal{Z}_{CG} = \sum_{N=0}^{\infty} \frac{\mu^{2N}}{N!} \sum_{\{\mathbf{q}\}}{}' \mathcal{Z}_N(\{\mathbf{q}\}|\beta). \tag{15}$$

Here $\mathcal{Z}_0$ is a partition function of free massless isovectorial boson field which corresponds to "spin waves" of XY -model.

This gives an embedding of the compact σ-models on T_G into noncompact generalized SG theories

$$\mathcal{Z}_{CG} = \int D\,\boldsymbol{\phi}\, e^{-\mathcal{S}_{eff}}, \quad \mathcal{S}_{eff} = \int \frac{1}{2\beta}(\partial\,\boldsymbol{\phi}\,)^2 - \mu^2 V(\,\boldsymbol{\phi}\,), \tag{16}$$

$$V(\,\boldsymbol{\phi}\,) = \sum_{\{\mathbf{q}\}} \exp i(\mathbf{q}\,\boldsymbol{\phi}\,).$$

where $\sum_{\{\mathbf{q}\}}$ goes over the set of minimal topological charges, and $\boldsymbol{\phi} \in \mathbb{R}^n$. The initial NSM correspond to some relation between parameters μ and β.

The account of vortices reduces the initial symmetry group N_G into discrete dual group $W_{G^*} \times L_q^{-1}$ (W_{G^*} is a dual Weyl group, L_q^{-1} is a periodicity lattice of potential V). This dual group generalizes the dual group $Z_2 \times \mathbb{Z}$ of XY-model.

Thus, in this semiclassical and long wavelength approximation

Compact theory on a torus T_G with *continuous* symmetry N_G appears equivalent (modulo $\mathcal{Z}_0$) to *noncompact* theory with periodic potential and an *infinite discrete* symmetry.

In case $\tau = ad$ the generalized SG theories can describe other systems with symmetry G broken to N_G [3].

IV. ADE LATTICES, TPT AND SYMMETRIES

The critical properties of the BKT type PT can be determined by renorm-group method [14, 21]. The new critical properties appear only in case, when each vector $\mathbf{q} \in \{\mathbf{q}\}$ can be represented as a sum of two other vectors. This condition concides with a definition of the root systems $\{\mathbf{r}\}$ of simple groups from series A, D, E or of the root sets of the even integer-valued lattices of $\mathbb{A}, \mathbb{D}, \mathbb{E}$ types. Moreover, the sets of minimal roots (and minimal weights) of all simple groups belong to four series of the integer-valued (in appropriate scale) lattices $\mathbb{A}, \mathbb{D}, \mathbb{E}, \mathbb{Z}$. Z_n is an example of the odd self-dual (or unimodular) lattices and contains a minimal vectors with norm equal 1, while the series A_n, D_n, E_n belong to the even lattices with minimal norm equal 2.

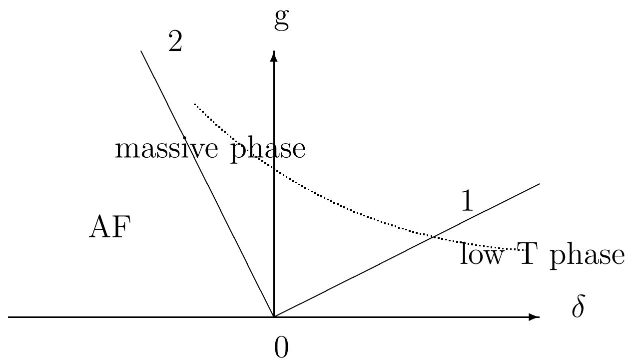
Each root set is characterised by the Coxeter number h_G

$$h_G = \frac{\text{(number of all roots)}}{\text{(rank of group)}},$$

$$h_{A_n} = n+1,\ h_{D_n} = 2(n-1),\ h_{E_{6,7,8}} = 12, 18, 30.$$

All coefficients of RG equations are expressed only through the Coxeter numbers h_G

The phase diagram have two separatrices with next declinations (u_1 corresponds to the phase separation line): $u_{1,2} = (1/\pi h_G, -1/2\pi)$.



The dashed line of the initial values corresponds to the initial σ-model. The critical exponent ν_G is inverse to the Lyapunov index of the separatrix 1 and equals

$$\nu_G = 2/(2 + h_G).$$

It can take the next values [3, 4]

G	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
ν_G	$\frac{2}{n+3}$	$\frac{1}{n}$	$\frac{1}{2}$	$\frac{1}{n}$	$\frac{2}{5}$	$\frac{1}{4}$	$\frac{1}{7}$	$\frac{1}{10}$	$\frac{1}{16}$

The critical properties of some systems with different symmetries can coincide due to coincidence of their h_G.

Low-T phase properties

The low-temperature phase is described by effective free field theory with a renomalized "temperature" $\bar{\beta}$, depending on initial values β_0. At the PT point (where $\bar{\beta} = \beta^* = 8\pi/r^2 = 4\pi$) an additional logarithmic factors, related with the "null charge" behaviour of the renormalized parameters on the critical separatrix (the phase separation line), can appear.

Free-like behaviour of the low-temperature phase (except logarithmic corrections at criticality) admits for its description the conformal field theories with integer central charges $C = n$, instead of PT points of two-dimensional systems with discrete symmetries, described by conformal theories with rational central charges [1, 8]. The BKT PT can be considered as the limiting case $k \to \infty$ (where k is a level) of the sequence of minimal conformal theories with $C = 1 - 6/(k+1)(k+2)$ [19]. Analogously, the TPT in σ-models on T_G are the limiting cases of unitary minimal conformal theories, connected with conformal W-algebras [9]. There exists a puzzling coincidence of ν_G with "screening" factor in formulae for central charges of the affine Lie algebras $\hat{\mathfrak{G}}$ [13] at level $k = 2$ (though T_G corresponds to $k = 1$)

$$C_k = \frac{k}{k + h_G} dimG$$

and of the coset realization of the minimal unitary conformal models [10] at level $k = 1$

$$C_k = r\left(1 - \frac{h_G(h_G+1)}{(k+h_G)(k+h_G+1)}\right).$$

Properties of massive phase

In this phase all additional vector charges will be shielded like in plasma. All excitations are massive. There is another enlargement of the isotopic symmetry of the initial NSM on separatrix 2. σ-model on T_G has at classical level two continuous symmetries: 1) scale (or conformal) symmetry, 2) isotopic global symmetry $N_G = T_G \times W_G$. Both symmetries are spontaneously broken in IR region by vortices. For this reason σ-model has in massive phase a finite correlation length $\xi \sim m^{-1}$, where m is some characteristic mass scale of the theory. This mass can depend on the coupling constant β.

On separatrix 2 one obtaines

$$m \sim \Lambda \exp(-1/2\pi g h_G), \quad \Lambda \sim a^{-1}.$$

This mass scale is defined only by $K_G \sim h_G$ (note that here $G = ADE$) and coincides in main approximation with those for chiral models on groups G [17, 16, 7] and for fermionic models with symmetry group G [7].

Mass scale of NSM on T_G coincides on separatrix 2 with that for all G-invariant theories, the chiral as well as fermionic (at least for $G = A, D, E$).

It means a possible restoration of the full isotopic symmetry group G in σ-models with symmetry group N_G in massive phase.

ADE classification

There are a number of other integer-valued lattices, which can serve as a lattice of topological charges. Their classification is not completed at present (except some low-dimensional cases) [6]. But, if one confines himself with NSM on tori $\mathcal{T}_L$ with integer-valued lattices of topological charges (their importance was noted above), then all possible types of critical behaviour will belong only to $ADEZ$ series. This conclusion follows from the Witt theorem, proving that the minimal vector sets of any integer-valued lattice must be a direct sum of the root systems [6]. But the last can be only of A, D, E, Z types. Consequently, all NS-models on tori $\mathcal{T}_L$ with integer-valued lattices of topological charges L_L^t with minimal norm equal 1 can have critical properties only of XY-model (or of Z^n lattice) type, while all NS-models on tori $\mathcal{T}_L$ with integer-valued lattices L_L^t with minimal norm equal 2 can have critical properties only of A, D, E lattice types. In this relation it is worth to note that analogous ADE classifications take place in the theory of singularities [] and in the string theory []. . In general case of integer-valued lattices the different components of the minimal vector sets can have different h_G. Then one can have in NSM on tori $\mathcal{T}_L$ with general integer-valued lattice L_L^t a series of PTP, taking place separately in each component (with critical properties, depending on h_G). For even self-dual lattices with minimal norm 2 all components must have the same Coxeter number [6], and, consequently, the PTP in all components take place simultaneously [5].

The TPT in NSM on degenerated tori can have an application to the description of partial space decompactification in string theories [5].

V. CONCLUSIONS

1. It is shown that one must consider deformed tori for obtaining the interacting vector topological charges.

2. Vector topological charges form a lattice and in some cases can reproduce all quantum numbers of the corresponding groups.

3. A sequence of approximately equivalent transformations of 2d models is constructed

$$\boxed{\begin{array}{c}\text{General GL Theory} \rightarrow \text{NS Model} \rightarrow \text{TopExcGas} \\ \rightarrow \text{General SG Theory}\end{array}}$$

It simplifies a problem and extracts all necessary long-wave properies of these theories! Here last theory has a pure group-theoretical structure and is universal for whole class of theories with the same symmetries.

4. All critical properties are classified by integer-valued lattices from series $\mathbb{A}, \mathbb{D}, \mathbb{E}, \mathbb{Z}$ and are characterised by the corresponding Coxeter numbers.

5. The possible scenarios of the dynamical enlargement of the initial internal symmetry groups is discussed.

6. Some applications of these TPT for cosmological and string theories is proposed.

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References

[1] Belavin A.A., Polyakov A.M., Zamolodchikov A.B., Nucl.Phys. **B241** (1984) 333.

[2] Berezinskii V.L., ZhETP **59** (1970) 907, **61** (1971) 1144.

[3] Bulgadaev S.A., Phys.Lett. **86A** (1981) 213; Theor.Math.Phys. **49** (1981) 7; Nucl.Phys. **B224** (1983) 349.

[4] Bulgadaev S.A., ZhETP Letters **63** (1996)a 780; 796; hep-th/9906091; hep-th/9901035.

[5] Bulgadaev S.A., hep-th/9811226, (1998).

[6] Conway G,H., Sloane N.J.A., Sphere Packings, Lattices and Groups, v.I,II. Springer-Verlag, 1988.

[7] Destri C., de Vega H.J., Preprint CERN-TH, 4895/87.

[8] Dotsenko Vl.S., Fateev V.A., Nucl.Phys.**B240** (1984) 312; **B251** (1985) 691; Nienhuis ,

[9] Fateev V.A., Lukyanov S.L., Int.Jour.Mod.Phys. **A13** (1988) 507.

[10] Goddard P., Kent A., Olive D., Phys.Lett.**B152** (1985) 88, Commun.Math.Phys. **103** (1986) 105.

[11] Green M., Schwarz J.H., Witten E., Superstrings Theory, Cambridge, 1988, vol.1,2.

[12] Jose J., Kadanoff L., Kirkpatrick S., Nelson D., Phys.Rev. **B16** (1977) 1217.

[13] Kac V.N., Infinite dimensional Lie algebras, Cambridge University Press, 1990.

[14] Kosterlitz J.M., Thouless J.P., J.Phys. **C6** (1973) 118; Kosterlitz J.M., J.Phys. **C7** (1974) 1046.

[15] Nelson D.R., Phys.Rev. **B18** (1978) 2318.

[16] Ogievetskii E., Wiegmann P.B., Phys.Lett. **B168** (1986) 360.

[17] Polyakov A.M., Gauge Fields and Strings, Harwood Academic Publishers, 1987.

[18] Popov V.N., unpublished (1971); Feynman integrals in quantum field theory and statistical mechanics, Moscow. Atomizdat. 1976.

[19] Reshetikhin N., Smirnov F., Commun.Math.Phys., **31** (1990) 157.

[20] Rice T.M., Phys.Rev. **140** (1966) 1889.

[21] Wiegmann P.B., J.Phys. **C11** (1978) 1583.